Kähler-Einstein Supermanifolds

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Abstract

In the first section we find that if a Kähler supermanifold satisfies Einstein’s
equation with cosmological constant and has Kähler potential of the form $K = A + Cθ\bar{θ}$, then the manifold obtained by setting the fermionic coordinate $θ$ to zero must have constant scalar curvature; this constant depends on the cosmological constant and the dimension of the manifold.

As a corollary, we find that every Kähler supermanifold with constant scalar curvature has a unique “superextension” with Kähler potential of the form $K = A + Cθ\bar{θ}$. We apply this result to show that the weighted super-projective space $\mathbb{W}_{1^n+1}^n$ satisfies Einstein’s equation–its base manifold is $\mathbb{C}P(1^n+1)$.

In the second section we investigate if the condition $R = 0$ imposes some constraint on the curvature of the base manifold of a class of supermanifolds, in analogy to [1]. We do find such a constraint:

$$\phi^i_\bar{j} = 2\Delta R_0 - R^2 + R^\bar{i}R_{ij}$$

where $\Delta φ = 0$. 
1 Introduction

A result of Roček and Wadhwa [1] was that if a Kähler supermanifold with one fermionic dimension has a Ricci tensor of zero, then the bosonic base manifold has zero Ricci scalar. Those calculations are actually a bit more general—they show that given any supermanifold whose Kähler potential can be written $K = A + C\theta\bar{\theta}$ for fermionic coordinate $\theta$, the supermanifold obtained by setting $\theta$ to zero has zero Ricci scalar. We generalize this result on Kähler supermanifolds in two ways.

In the first section we find that if a Kähler supermanifold satisfies Einstein’s equation with cosmological constant and has Kähler potential of the form $K = A + C\theta\bar{\theta}$, then the manifold obtained by setting the fermionic coordinate $\theta$ to zero must have constant scalar curvature; this constant depends on the cosmological constant and the dimension of the manifold.

As a corollary, we find that every Kähler supermanifold with constant scalar curvature has a unique “superextension” with Kähler potential of the form $K = A + C\theta\bar{\theta}$. We apply this result to show that the weighted super-projective space $\mathbb{WSP}(1,\ldots,1|n)$ satisfies Einstein’s equation—its base manifold is $\mathbb{CP}(1,\ldots,1)$. We apply this result to show that the weighted super-projective space $\mathbb{WSP}(1,\ldots,1|n)$ satisfies Einstein’s equation—its base manifold is $\mathbb{CP}(1,\ldots,1)$. We apply this result to show that the weighted super-projective space $\mathbb{WSP}(1,\ldots,1|n)$ satisfies Einstein’s equation—its base manifold is $\mathbb{CP}(1,\ldots,1)$. We apply this result to show that the weighted super-projective space $\mathbb{WSP}(1,\ldots,1|n)$ satisfies Einstein’s equation—its base manifold is $\mathbb{CP}(1,\ldots,1)$. We apply this result to show that the weighted super-projective space $\mathbb{WSP}(1,\ldots,1|n)$ satisfies Einstein’s equation—its base manifold is $\mathbb{CP}(1,\ldots,1)$.

In the second section we investigate if the condition $R = 0$ imposes some constraint on the curvature of the base manifold of a class of supermanifolds, in analogy to [1]. We do find such a constraint:

$$\phi^{ij}\phi_{ij} = 2\Delta R_0 - R^2 + R^{ij}R_{ij}$$

where $\Delta \phi = 0$.

This calculation can be thought of as the second step in a hierarchy of equations relating the curvature of a supermanifold to the curvature of its lower-dimensional base supermanifold. Zero Ricci tensor leads to zero Ricci scalar, and zero Ricci scalar leads to the differential equation above.

2 Cosmological constant

In this section we will show that if a Kähler supermanifold satisfies Einstein’s equation with cosmological constant and has Kähler potential of the form

$$K = A + C\theta\bar{\theta}$$

(2.1)
then the manifold obtained by setting the fermionic coordinate \( \theta \) to zero must have constant scalar curvature; this constant depends on the cosmological constant and the dimension of the manifold. Note that the most general form the Kähler potential can have is

\[
K = A + B\bar{\theta} + \theta\bar{B} + C\theta \bar{\theta}
\]  

(2.2)

Here, so that the Kähler potential is real and commuting, \( A \) and \( C \) are real and commuting, while \( B \) is anticommuting. In the case that \( \theta \) is the only fermionic coordinate, all Kähler potentials have the form (2.1), so (2.1) is not an additional condition at all.

In these supermanifolds we can globally set the coordinate \( \theta = 0 \) and obtain a lower-dimensional supermanifold whose Kähler potential is \( A \).

Using the following conventions

\[
R_{ij} = -\partial_i \partial_j \ln \det g_{ij}
\]  

(2.3)

\[
R = -g^{ij} \partial_i \partial_j \ln \det g_{ij}
\]  

(2.4)

which rescale the standard definitions of \( R \) and \( R_{i\bar{j}} \) by factors of 2 and 4, respectively, Einstein’s equation reads

\[
R_{i\bar{j}} - R g_{i\bar{j}} + \Lambda g_{i\bar{j}} = 0
\]  

(2.5)

Taking the supertrace of this equation

\[
0 = g^{i\bar{j}}(R_{i\bar{j}} - R g_{i\bar{j}} + \Lambda g_{i\bar{j}}) = R + (\Lambda - R)g^{i\bar{j}} g_{i\bar{j}}
\]  

(2.6)

The indices \( I, \bar{J} \) index over all coordinates including \( \theta, \bar{\theta} \); on the other hand, the indices \( i, \bar{j} \) (used later in this section) do not index over \( \theta, \bar{\theta} \).

\( g^{i\bar{j}} g_{i\bar{j}} \) is equal to the superdimension \( D = D_{\text{bose}} - D_{\text{fermi}} \), the number of complex bosonic dimensions minus the number of complex fermionic dimensions. Solving for \( R \), we get

\[
R = \frac{\Lambda D}{D - 1}
\]  

(2.7)

Thus the scalar curvature of the whole manifold is constant. Einstein’s equation becomes

\[
R_{i\bar{j}} = -\Upsilon g_{i\bar{j}}
\]  

(2.8)

where \( \Upsilon = \frac{\Lambda}{D - 1} \). Using (2.3) we get

\[
\partial_i \partial_{\bar{j}} \ln \det g + \Upsilon \partial_i \partial_{\bar{j}} K = 0
\]

\[
\partial_i \partial_{\bar{j}} [\ln \det g + \Upsilon K] = 0
\]  

(2.9)
This implies that we can perform a holomorphic coordinate transformation\(^1\) so that in our new coordinates, we have the equation

\[
sdet g = e^{-\Upsilon K}
\]  
(2.10)

Substituting in (2.1) and taking the Taylor expansion,

\[
sdet g = e^{-\Upsilon A}(1 + C\bar{\theta}\bar{\theta})
\]  
(2.11)

Now that we have found an equivalent form of Einstein’s equation on these Kähler-Einstein manifolds, we will calculate sdet \(g\).

\[
g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K = \partial_i \partial_{\bar{j}} (A + C\bar{\theta}\bar{\theta})
\]  
(2.12)

\[
= \begin{pmatrix}
 A_{i\bar{j}} + \bar{\theta}\bar{\theta}C_{i\bar{j}} & C_i\bar{\theta} \\
 C_{\bar{j}}\bar{\theta} & C
\end{pmatrix}
\]  
(2.13)

Then

\[
sdet(g) = \frac{1}{C} sdet(A_{i\bar{j}} + \bar{\theta}\bar{\theta}(C_{i\bar{j}} - \frac{C_i C_{\bar{j}}}{C}))
\]  
(2.14)

\[
A_{i\bar{j}} + \bar{\theta}\bar{\theta}(C_{i\bar{j}} - \frac{C_i C_{\bar{j}}}{C}) = A_{i\bar{j}} + \bar{\theta}(C_{i\bar{j}} - \frac{C_i C_{\bar{j}}}{C}) - \frac{1}{2}(C_{i\bar{j}} - \frac{C_i C_{\bar{j}}}{C}) (A_{i\bar{j}} - \frac{C_i C_{\bar{j}}}{C})
\]  
(2.15)

Therefore,

\[
sdet(g) = \frac{1}{C} sdet(A_{i\bar{j}}) sdet(\delta_{i\bar{j}} - \frac{C_i C_{\bar{j}}}{C})\bar{\theta}\bar{\theta}
\]  
(2.16)

Taking the Taylor expansion around the identity, all order terms in \(\bar{\theta}\bar{\theta}\) vanish and we are left with

\[
sdet(g) = \frac{1}{C} sdet(A_{i\bar{j}})(1 + \bar{\theta}(C_{i\bar{j}} - \frac{C_i C_{\bar{j}}}{C})\bar{\theta}\bar{\theta})
\]  
(2.17)

Comparing this equation to (2.10), the bosonic part gives us

\[
e^{-\Upsilon A} = \frac{1}{C} sdet(A_{i\bar{j}})
\]  
(2.18)

The fermionic part gives us

\[
-e^{-\Upsilon A}\Upsilon C = \frac{1}{C} sdet(A_{i\bar{j}})A_{i\bar{j}}(C_{i\bar{j}} - \frac{C_i C_{\bar{j}}}{C})
\]  
(2.19)

\(^1\)If \(i\partial_i F = 0\) for all \(I, J\), then \(F(z, \bar{z}) = f(z) + g(\bar{z})\)—it is the sum of a holomorphic function (power series in the holomorphic coordinates) and an antiholomorphic function (power series in the antiholomorphic coordinates.) If \(F\) is real, then \(g(\bar{z})\) must be the complex conjugate of \(f(z)\), so \(F(z, \bar{z}) = f(z) + \bar{f}(\bar{z})\). (2.9) becomes

\[
\ln sdet g = -\Upsilon K + f(z) + \bar{f}(\bar{z})
\]

\[
sdet g = e^{-\Upsilon K} e^{f(z) + \bar{f}(\bar{z})} = e^{-\Upsilon K |e^{f(z)}|^2}
\]

The factor of \(|e^{f(z)}|^2\) can be canceled by the Jacobian of a suitable holomorphic coordinate transformation.
(2.18) gives us

\[- \Upsilon C = A^{\bar{i}} (C_{,ij} - \frac{C_i C_j}{C}) \quad (2.20)\]

\[- \Upsilon = g^{\bar{i}j} \partial_i \partial_j \ln(C) \quad (2.21)\]

Substituting in (2.1) gives us

\[- \Upsilon = g^{\bar{i}j} \partial_i \partial_j \ln(\frac{s\det A_{,ij}}{e^{-\Upsilon A}}) \quad (2.22)\]

\[- \Upsilon = g^{\bar{i}j} \partial_i \partial_j \ln(\text{sdet}(A_{,ij}) - g^{\bar{i}j} \partial_i \partial_j (-\Upsilon A)) \quad (2.23)\]

\[- \Upsilon = g^{\bar{i}j} \partial_i \partial_j \ln(\text{sdet}(A_{,ij}) + \Upsilon g^{\bar{i}j} g_{ij}) \quad (2.24)\]

\[- \Upsilon = -R_0 + \Upsilon D_0 \quad (2.25)\]

\[R_0 = (D_0 + 1) \Upsilon = (D + 2) \Upsilon = -\Lambda \frac{D + 2}{D - 1} \quad (2.26)\]

Here $R_0$ is the Ricci scalar of the submanifold defined by $\theta = 0$, and $D_0$ is its complex superdimension.

We can also carry out these steps in the opposite direction: we can take any Kähler manifold with constant Ricci curvature and add a fermionic dimension parameterized by $\theta$ and $\bar{\theta}$. In the following example, we find the “superextension” of the complex projective plane $\mathbb{C}P^n$.

**Superextension of $\mathbb{C}P^n$**

A Kähler supermanifold satisfies Einstein’s vacuum equations if (2.11) holds, which holds if (2.18) and (2.26) both hold. (2.26) only concerns the base manifold: it says that the base manifold must have constant scalar curvature. The Fubini-Study metric of $\mathbb{C}P^n$, which is “round” because it leaves all points equivalent, is given by Kähler potential

\[K = \ln(1 + z^k \bar{z}_k) \quad (2.27)\]

Because of its symmetry, this manifold has constant scalar curvature. (2.18) gives us

\[C = \frac{\text{det}(g_{ij})}{e^{-\Upsilon A}} \quad (2.28)\]

Now we will calculate the determinant of $g$:

\[g_{ij} = K_{,ij} = \frac{\delta_{ij}}{1 + z^k \bar{z}_k} - \frac{z_i z_j}{(1 + z^k \bar{z}_k)^2} \quad (2.29)\]
One eigenvector of this matrix is $z^i = (z^1, z^2, \ldots, z^n)$, since

$$z^i g_{ij} = z^i \left( \delta_{ij} - \frac{z_i z_j}{(1 + z^k z_k)^2} \right) = z_j - \frac{z^i z_i z_j}{(1 + z^k z_k)^2}$$

$$= \frac{z_j + z^i z_k z^j - z^i z_i z^j}{(1 + z^k z_k)^2} = \frac{z_j}{(1 + z^k z_k)^2}$$

This eigenvalue is $\frac{1}{(1 + z^k z_k)^2}$

Choose vector $x^i$ so that $x^i z_i = 0$. $x^i$ is an eigenvector, since

$$x^i g_{ij} = x^i \left( \delta_{ij} - \frac{z_i z_j}{(1 + z^k z_k)^2} \right) = x_j - \frac{x^i z_i z_j}{(1 + z^k z_k)^2}$$

$$= \frac{x_j}{1 + z^k z_k}$$

This eigenvalue is $\frac{1}{1 + z^k z_k}$, and it has a degeneracy of $n - 1$.

The determinant is the product of the eigenvalues, so

$$\det g = \frac{1}{(1 + z^k z_k)^{n+1}}$$

Substituting in (2.34) into (2.28),

$$C = \frac{\frac{1}{(1 + z^k z_k)^{n+1}}}{\frac{1}{1 + z^k z_k}} = \frac{1}{(1 + z^k z_k)^{n}}$$

The total Kähler potential is

$$\ln(1 + z^k z_k) + \frac{\theta \bar{\theta}}{(1 + z^k z_k)^{n}}$$

The resulting supermanifold is weighted projective space: $\mathbb{WSP}(1, \ldots, 1 | n)$.

### 3 Zero Ricci scalar

Again we consider manifolds with a Kähler potential of the following form:

$$K = A + C\theta \bar{\theta}$$

(3.1)
Through a long calculation, we will write the equation $R = 0$ in terms of $A$ and $C$. Just as in [1], the bosonic part of this equation puts a constraint on $A$ and $C$, and this allows us to write the fermionic part of the equation in terms of $A$, the Kähler potential of the base manifold. What emerges is a purely geometric equation describing the curvature of the base manifold:

$$\phi^{ji} \phi_{,ij} = 2\Delta R_0 - R^2 + R^{ji} R_{ij}$$  \hspace{1cm} (3.2)

where $\Delta_0 \phi = 0$.

$$g_{ij} = \partial_i \partial_j K = \left( A_{ij} + C_{ij} \theta \bar{\theta} \begin{array}{c} \bar{C} \theta \\ C \theta \end{array} \right)$$  \hspace{1cm} (3.3)

Using the formula for the inverse of a supermatrix $^1$, and using the fact that $(\ln C)_{ij} = \frac{C_{ij}}{C} - \frac{C_i C_j}{C^2}$, we get

$$g^{ji} = \left( [A_{ij} + \theta \bar{\theta} C (\ln C)_{ij}]^{-1} \begin{array}{cc} - (A^{-1})^{ji} C_i \theta \\ \bar{C} \theta (A^{-1})^{ji} C_i \end{array} \right)$$  \hspace{1cm} (3.5)

$$= \left( (A^{-1})^{ji} - \theta \bar{\theta} C (A^{-1})^{jk} (\ln C)_{ki} (A^{-1})^{li} \begin{array}{cc} - (A^{-1})^{ji} C_i \theta \\ \bar{C} \theta C_i^{-1} \end{array} \right)$$  \hspace{1cm} (3.6)

Since $A_{ji}$ is the metric of the base manifold, we can apply the notational convention that the inverse metric $(A^{-1})^{ji}$ raises indices of tensors.

$$g^{ji} = \left( A^{ji} - \theta \bar{\theta} C (\ln C)^{ji} \begin{array}{cc} \frac{C_j \theta}{C} \\ - \bar{C} \theta C_i^{-1} \end{array} \right)$$  \hspace{1cm} (3.7)

The Laplacian on a Kähler supermanifold is defined as the operator $\Delta = g^{ji} \partial_i \partial_j$. We find

$$\Delta = g^{ji} \partial_j \partial_i + g^{\theta \bar{\theta}} \partial_\theta \partial_\bar{\theta} + g^{\theta i} \partial_i \partial_\bar{\theta} + g^{\theta \bar{\theta}} \partial_\theta \partial_\theta$$

$$= \Delta_0 - \theta \bar{\theta} C (\ln C)^{ji} \partial_j \partial_i - \bar{C} \theta \partial_\theta \partial_\bar{\theta} C_i^{-1} \partial_i + C^{-1} \partial_\bar{\theta} \partial_\theta - \theta \bar{\theta} C_i C_j \partial_\theta \partial_\bar{\theta}$$  \hspace{1cm} (3.8)

$^1$The inverse of a supermatrix in block form, where the blocks can have both bosonic and fermionic parts, is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$  \hspace{1cm} (3.4)
Taking the superdeterminant \(^2\) of the metric

\[
\text{sdet} \ g_{IJ} = \frac{1}{C} \text{sdet}(A_{ij} + C_{i\bar{j}}\theta\bar{\theta} - \frac{C_{i\bar{j}}\bar{C}}{C}\theta\bar{\theta}) \quad (3.11)
\]

\[
= \frac{1}{C} \text{sdet}(A_{i\bar{j}}) \text{sdet}(\delta_{i}^{\bar{j}} + CA^{lm}(\ln C)_{,mk}\theta\bar{\theta}) \quad (3.12)
\]

\[
= \frac{1}{C} \text{sdet}(A_{i\bar{j}})(1 + C(\ln C)_{,i}\bar{\theta}\theta) \quad (3.13)
\]

\[
= \frac{1}{C} \text{sdet}(A_{i\bar{j}})(1 + C\Delta_0 \ln C\theta\bar{\theta}) \quad (3.14)
\]

\[
\ln \text{sdet} \ g_{IJ} = -\ln C + \ln \text{sdet} A_{ij} + \theta\bar{\theta}C\Delta_0 \ln C \quad (3.15)
\]

Substituting (3.9) and (3.15) into equation (2.4) for the Ricci scalar, we get

\[
0 = \Delta \ln \text{sdet} g_{IJ} = -\Delta_0 \ln C + \Delta_0 \ln \text{sdet} A_{ij} - \Delta_0 \ln C + \theta\bar{\theta}[\Delta_0(C\Delta_0 \ln C) + C(\ln C)_{,i}\bar{\theta}\theta - 2(\ln C)_{,i}(\ln \text{sdet} A)_{,i} + (\ln C)_{,i}(\ln C)_{,i}C\Delta_0 \ln C] \quad (3.16a)
\]

\[
\Delta_0(C\Delta_0 \ln C) = C\Delta_0\Delta_0 \ln C + (\Delta_0 C)(\Delta_0 \ln C) + 2C_{,i}(\Delta_0 \ln C)^{,i} = C\Delta_0\Delta_0 \ln C + C\frac{C_{,i}^{,i}}{C}(\Delta_0 \ln C) + 2C_{,i}(\Delta_0 \ln C)^{,i} \quad (3.20a)
\]

(3.16c) becomes

\[
C(\ln C)_{,i}\bar{j}(\ln C)_{,i}j - C(\ln C)_{,i}\bar{j}(\ln \text{sdet} A)_{,i}j = C(\ln C)_{,i}\bar{j}(\ln C - \ln \text{sdet} A)_{,i}j \quad (3.21)
\]

\[
= C\left(\frac{1}{2} \ln \text{sdet} A + \frac{1}{2}\phi\right)_{,i}\bar{j}(\ln C - \ln \text{sdet} A + \frac{1}{2}\phi)_{,i}j \quad (3.22)
\]

\[
= -\frac{1}{4}(\ln \text{sdet} A)_{,i}^{,j}(\ln \text{sdet} A)_{,i}j + \frac{\phi_{,i}^{,j}\phi_{,i}j}{4} \quad (3.23)
\]

\(^2\)The superdeterminant of a block matrix with fermionic block \(D\) is

\[
\text{sdet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{\det(D)} \text{sdet}(A - BD^{-1}C) \quad (3.10)
\]
(3.16d) becomes
\[-2(\ln C)^{i}(C \Delta \ln C, i) = -2C(\ln C)^{i}(\Delta_{0} \ln C, i) - 2(\ln C)^{i}C_{i} \Delta_{0} \ln C =
\]
\[-2C^{i}(\Delta_{0} \ln C)_{i} \quad (3.24a)
\]
\[-2C^{i}C_{i} \Delta_{0} \ln C \quad (3.24b)
\]
\[-C \frac{C_{i} \Delta_{0} \ln C}{C^{2}} \quad (3.24c)
\]
The fermionic part of $R$ is $\text{(3.20a) + (3.20b) + (3.20c) + (3.23) + (3.24a) + (3.24b) + (3.24c) + (3.16e)}$. Comparing these terms, we find $\text{(3.20b) + (3.24c) = } C(\Delta_{0} \ln C)^{2}$, $\text{(3.20c) + (3.24a) = 0}$, and $\text{(3.16e) + (3.24b) = 0}$. Thus
\[
0 = C(\Delta_{0} \Delta_{0} \ln C + (\Delta_{0} \ln C)^{2} - \frac{1}{4}(\ln \text{sdet } A)\tilde{\phi}_{i j}(\ln \text{sdet } A)_{i j} + \frac{\phi^{j i} \phi_{j i}}{4}) \quad (3.25)
\]
\[
0 = \frac{1}{2} \Delta_{0} \Delta_{0} \ln \text{sdet } A + \frac{1}{4}(\Delta_{0} \ln \text{sdet } A)^{2} - \frac{1}{4}(\ln \text{sdet } A)\tilde{\phi}_{i j}(\ln \text{sdet } A)_{i j} + \frac{\phi^{j i} \phi_{j i}}{4} \quad (3.26)
\]
\[
= \frac{1}{2} \Delta_{0} R + \frac{1}{4} R^{2} - \frac{1}{4} R\tilde{\phi}_{i j} + \frac{\phi^{j i} \phi_{j i}}{4} \quad (3.27)
\]
Finally,
\[
\phi^{j i} \phi_{j i} = 2\Delta R_{0} - R^{2} + R\tilde{\phi}_{i j} \quad (3.28)
\]
This is a purely geometric equation as we wanted. The term $\phi^{j i} \phi_{j i}$ is restricted by the condition $\Delta_{0} \phi = 0$. If the manifold is compact and bosonic, this condition implies that $\phi$ is constant, since constant functions are the only ones with zero Laplacian everywhere. In the case that the base manifold is one dimensional and bosonic, $-R^{2} + R\tilde{\phi}_{i j} = 0$, and $\Delta R_{0} = 0$, which is consistent with the result of [1].

4 Conclusion

We have studied the relationship between the curvature of a Kähler-Einstein manifold with Kähler potential $K = A + C\theta\bar{\theta}$ and the curvature of the base manifold. The most immediate generalization of these calculations, as well as the calculations of [1], is to allow for the more general Kähler potential $K = A + B\bar{\theta} + \theta\bar{B} + C\theta\bar{\theta}$. However, it is difficult to work in this general case, and new methods may be needed. The problem is especially significant because a Kähler potential of the form $K = A + B\bar{\theta} + \theta\bar{B} + C\theta\bar{\theta}$ in general can not be put into the simpler form $K' = A + C\theta'\bar{\theta}$ by a holomorphic coordinate transformation, even when there are only two fermionic coordinates.
References


